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THE BUCKLING OF PARALLEL SIMPLY SUPPORTED TENSION AND COMPRESSION MEMBERS CONNECTED BY ELASTIC DEFLECTIONAL SPRINGS

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SUMMARY

An investigation of the problem of the buckling of parallel simply supported tension and compression members connected by equally stiff and equally spaced elastic deflectional springs is made as an approximation to the problem of the effect of finite stiffness of ribs and tension surface on the buckling load of the compression surface of a wing. Charts relating compressive buckling load, deflectional spring stiffness, and the ratio of the flexural stiffness of the members – for the case of equal tension and compression loads – are given for tension and compression members having two, three, four, and an infinite number of spans.

INTRODUCTION

In the design of aircraft structures, the calculation of the compressive buckling load of the surface of a stressed-skin wing is important. For simplicity, the tension surface of the wing can be assumed to be equivalent to a rigid foundation and the shear webs and ribs can be assumed to be equivalent to rigid supports that divide the compression surface into small panels so that the compressive buckling load of the wing surface is the buckling load of the small panels. Actually, however, the shear webs and ribs and the tension surface have finite stiffness; therefore, the buckling load of the compression surface is reduced because of the deflection of the supports.

The effect of finite rib stiffness on the compressive buckling load of a wing surface has been investigated in a number of papers. (See references 1 to 6.) In the present paper an approximation to the problem of considering the effect of the finite stiffness of both the ribs and the tension surface on the buckling load of the compression surface of a wing is made by investigating the buckling of parallel simply supported tension and compression members connected by equally stiff and equally spaced elastic deflectional springs. (See fig. 1.) An exact Rayleigh-Ritz analysis of this problem is given in the appendix.

SYMBOLS

x	distance parallel to members
y_C	deflection of compression member
y_T	deflection of tension member
N	number of spans
L	length between springs
$(EI)_C$	flexural stiffness of compression member
$(EI)_T$	flexural stiffness of tension number
$r = \frac{(EI)_T}{(EI)_C}$	
P_C	critical compressive load
P_T	load applied to tension member
$j_C = \sqrt{\frac{(EI)_C}{P_C}}$	
$j_T = \sqrt{\frac{(EI)_T}{P_T}}$	
L/j_C	nondimensional buckling-load parameter $\left(\sqrt{\frac{P_C L^2}{(EI)_C}}\right)$
L/j_T	nondimensional tension-load parameter $\left(\sqrt{\frac{P_T L^2}{(EI)_T}}\right)$
C	deflection spring constant, force per unit deflection
S	nondimensional deflectional-spring-stiffness parameter $\left(\frac{CL^3}{(EI)_C}\right)$

m, n, s	integers
k	integer defining location of a spring
q	number of buckles

RESULTS AND DISCUSSION

Nondimensional charts are presented as figures 2 to 5 for the buckling load of parallel simply supported tension and compression members connected by elastic deflectional springs, for members with two, three, four, and an infinite number of spans and for equal tension and compression loads. The buckling load can be obtained from these figures when the flexural stiffness of the tension and compression members and the deflectional spring stiffness are known. The curves were obtained from the exact stability equations derived by the Rayleigh-Ritz energy method in the appendix for the more general case of unequal tension and compression loads.

The figures show that the effect of finite flexural stiffness of the tension member is to increase appreciably the deflectional spring stiffness necessary to attain a given buckling load above the stiffness required when the tension member is infinitely stiff. The maximum load attainable is the load for Euler buckling of the compression member with nodes at the deflectional-spring supports.

The discontinuities of slope of the curves of figures 2 to 4 correspond to sudden changes in the buckling mode. Tension and compression members having two spans, for instance, buckle first with one buckle until a limiting value of spring stiffness is reached then buckle in two half-waves with a node at the spring support. The curves for members having an infinite number of spans (fig. 5) are smooth because the buckling mode changes continuously. Curves for the case of a tension member with infinite flexural stiffness are identical with the curves of reference 7 for columns on deflectional springs alone.

In each of the figures the horizontal line is the curve for Euler buckling of the compression member with nodes at the spring supports. Because the supports do not deflect, the load is independent of their spring stiffness. The value of the spring stiffness at the intersection of the horizontal line and the curves for buckling of the members with deflection of the spring supports is therefore the maximum value that is needed in any design because larger values of spring stiffness do not increase the buckling load.

ILLUSTRATIVE EXAMPLE

Two simply supported members are connected by three equally spaced intermediate elastic springs and are restrained against deflecting out of the plane of the springs. Each member consists of a 5-inch by 5-inch by $\frac{7}{16}$-inch steel angle having a moment of inertia of 10 inches^4 and a length between spring supports of 120 inches. The constant of the intermediate springs is 8000 pounds per inch. One member is to carry a compressive load and the other is to carry a tensile load of equal magnitude. In order to determine the loads capable of being carried by the existing structure, the following procedure is used:

By use of the curve of figure 4 for $\frac{(EI)_T}{(EI)_C} = 1$ and a value of the deflectional-spring-stiffness parameter

$$\frac{CL^3}{(EI)_C} = \frac{8000 \times 120^3}{30 \times 10^6 \times 10}$$

$$= 46.1$$

the buckling-load parameter is found to be

$$\frac{P_C L^2}{(EI)_C} = 8.9$$

The equal tension and compression loads that can be carried by the structure are therefore equal to

$$P_C = 8.9 \frac{30 \times 10^6 \times 10}{120^2}$$

$$= 185{,}400 \text{ pounds}$$

If the tension member were infinitely stiff, the buckling load of the compression member would be increased to the load for Euler buckling of the column between spring supports

$$P_C = \frac{\pi^2}{8.9} 185,400$$

$$= 205,600 \text{ pounds}$$

and would remain at that value if the spring constant were reduced to a value as low as

$$C = \frac{33.7}{46.1} 8000$$

$$= 5850 \text{ pounds per inch}$$

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APPENDIX

DERIVATION OF THE STABILITY CRITERIONS

A column, subjected to a compressive load P_C, is stabilized by equally stiff and equally spaced elastic deflectional springs that rest on a member subjected to a tension load P_T. (See fig. 1.) When the column buckles the structure deflects as a unit because the buckling of the compression member, in general, causes the supporting springs either to compress or elongate; thus loads are transmitted to the tension member, and it is also caused to deflect. The column buckling load is found by choosing Fourier series to represent the deflection curves of the tension and compression members, expressing the energy of the structure in terms of the unknown Fourier coefficients, and minimizing the energy with respect to these coefficients. An infinite set of equations is obtained and is solved by the method used in reference 7 in which the buckling load of columns on equally spaced deflectional and rotational springs was obtained.

Energy Expressions

Let the deflection curve of the buckled column be denoted by the Fourier series

$$y_C = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{NL} \tag{A1}$$

and the deflection curve of the tension member by

$$y_T = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{NL} \tag{A2}$$

Then, the energy of the system consists of the following components: The bending energy stored in the buckled column

$$V_C = \frac{(EI)_C}{2} \int_0^{NL} \left(\frac{d^2 y_C}{dx^2}\right)^2 dx$$

$$= \frac{\pi^4}{4} \frac{(EI)_C}{(NL)^3} \sum_{n=1}^{\infty} n^4 {a_n}^2 \tag{A3}$$

and the bending energy stored in the tension member

$$V_T = \frac{(EI)_T}{2} \int_0^{NL} \left(\frac{d^2 y_T}{dx^2}\right)^2 dx$$

$$= \frac{\pi^4}{4} \frac{(EI)_T}{(NL)^3} \sum_{n=1}^{\infty} n^4 b_n^{\,2} \tag{A4}$$

The energy stored in the deflectional springs is

$$V_S = \sum_{k=1}^{N-1} \frac{C}{2} \left[\left(y_C - y_T\right)_{x=kL}\right]^2$$

$$= \frac{C}{2} \sum_{k=1}^{N-1} \left[\sum_{n=1}^{\infty} \left(a_n - b_n\right) \sin \frac{n\pi k}{N}\right]^2 \tag{A5}$$

The work done by the compressive load in moving the two ends of the column together is

$$W_C = \frac{P_C}{2} \int_0^{NL} \left(\frac{dy_C}{dx}\right)^2 dx$$

$$= \frac{\pi^2}{4} \frac{P_C}{NL} \sum_{n=1}^{\infty} n^2 a_n^{\,2} \tag{A6}$$

and that done by the tension load is

$$W_T = -\frac{P_T}{2} \int_0^{NL} \left(\frac{dy_T}{dx}\right)^2 dx$$

$$= -\frac{\pi^2}{4} \frac{P_T}{NL} \sum_{n=1}^{\infty} n^2 b_n^{\,2} \tag{A7}$$

The work done by the tension load is negative inasmuch as the direction of the shortening of the tension member is opposite that of the load.

The potential energy of the system is equal to

$$F = V_C + V_T + V_S - W_C - W_T$$

which, from equations (A3) to (A7), is, after simplification,

$$F = \frac{\pi^4}{4}\frac{(EI)_C}{(NL)^3}\left\{\sum_{n=1}^{\infty}\left[n^4 - \left(\frac{NL}{\pi j_C}\right)^2 n^2\right]a_n{}^2 + r\sum_{n=1}^{\infty}\left[n^4 + \left(\frac{NL}{\pi j_T}\right)^2 n^2\right]b_n{}^2\right.$$

$$\left. + \frac{2N^3S}{\pi^4}\sum_{k=1}^{N-1}\left[\sum_{n=1}^{\infty}(a_n - b_n)\sin\frac{n\pi k}{N}\right]^2\right\} \tag{A8}$$

Minimization

The buckling load may be found by minimizing F with respect to the undetermined Fourier coefficients a_n and b_n. Then

$$\frac{\partial F}{\partial a_n} = 0$$

$$= \left[n^4 - \left(\frac{NL}{\pi j_C}\right)^2 n^2\right]a_n + \frac{2N^3S}{\pi^4}\sum_{m=1}^{\infty}(a_m - b_m)\sum_{k=1}^{N-1}\sin\frac{n\pi k}{N}\sin\frac{m\pi k}{N} \tag{A9}$$

$$(n = 1, 2, 3, \ldots)$$

and

$$\frac{\partial F}{\partial b_n} = 0$$

$$= r\left[n^4 + \left(\frac{NL}{\pi j_C}\right)^2 n^2\right]b_n - \frac{2N^3S}{\pi^4}\sum_{m=1}^{\infty}(a_m - b_m)\sum_{k=1}^{N-1}\sin\frac{n\pi k}{N}\sin\frac{m\pi k}{N} \tag{A10}$$

$$(n = 1, 2, 3, \ldots)$$

Equations (A9) and (A10) may be combined to give one set of equations with the coefficients $(a_n - b_n)$ as the unknowns:

$$(a_n - b_n) + \frac{2N^3 S}{\pi^4}\left\{\frac{1}{n^4 - \left(\frac{NL}{\pi j_C}\right)^2 n^2} + \frac{1}{r\left[n^4 + \left(\frac{NL}{\pi j_T}\right)^2 n^2\right]}\right\}\sum_{m=1}^{\infty}(a_m - b_m)\sum_{k=1}^{N-1}\sin\frac{n\pi k}{N}\sin\frac{m\pi k}{N} = 0 \qquad \text{(A11)}$$

$$(n = 1, 2, 3, \ldots)$$

except when $a_n = b_n$, in which case

$$n^4 - \left(\frac{NL}{\pi j_C}\right)^2 n^2 = n^4 + \left(\frac{NL}{\pi j_T}\right)^2 n^2 = 0 \qquad \text{(A12)}$$

$$(n = 1, 2, 3, \ldots)$$

Series-Form Stability Criterions

The stability criterions may be obtained from equations (A11) by the application of the method of solution of reference 7, simplified a great deal by the omission of rotational springs. The criterions so obtained are

$$\frac{\pi^4}{S} - \sum_{s=0}^{\infty}\left\{\frac{1}{\left(2s + \frac{q}{N}\right)^2\left(\frac{L}{j_C}\right)^2 - \left(2s + \frac{q}{N}\right)^4} + \frac{1}{\left[2(s + 1) - \frac{q}{N}\right]^2\left(\frac{L}{j_C}\right)^2 - \left[2(s + 1) - \frac{q}{N}\right]^4}\right\}$$

$$+ \frac{1}{r}\sum_{s=0}^{\infty}\left\{\frac{1}{\left(2s + \frac{q}{N}\right)^2\left(\frac{L}{j_T}\right)^2 + \left(2s + \frac{q}{N}\right)^4} + \frac{1}{\left[2(s + 1) - \frac{q}{N}\right]^2\left(\frac{L}{j_T}\right)^2 + \left[2(s + 1) - \frac{q}{N}\right]^4}\right\} = 0 \qquad \text{(A13)}$$

$$(q = 1, 2, \ldots N - 1)$$

and

$$\frac{L}{j_C} = \pi \qquad \text{(A14)}$$

$$(q = N)$$

However, when

$$\left(\frac{L}{j_C}\right)^2 \mp -\left(\frac{L}{j_T}\right)^2 = \left(\frac{\pi}{N}\right)^2 \qquad \text{(A15)}$$

both members have equal deflections and, therefore, the springs are not compressed and the spring stiffness is indeterminate.

Equations (A13) correspond to buckling in q buckles provided that q is less than the number of spans. Equation (A14) is the Euler buckling load for buckling of the column in N waves with a node at each spring support. Equations (A15), which were obtained from equations (A12), correspond to the case in which both members are subjected to their respective Euler buckling loads.

Closed-Form Stability Criterions

The series forms of the stability criterions (equations (A13)) may be put into closed form, and the result is

$$\frac{1}{S} - \left[\frac{1}{2\left(\frac{L}{j_C}\right)^2\left(1 - \cos \pi\frac{q}{N}\right)} + \frac{\sin \frac{L}{j_C}}{2\left(\frac{L}{j_C}\right)^3\left(\cos \pi\frac{q}{N} - \cos \frac{L}{j_C}\right)}\right]$$

$$+ \frac{1}{r}\left[\frac{1}{2\left(\frac{L}{j_T}\right)^2\left(1 - \cos \pi\frac{q}{N}\right)} + \frac{\sinh \frac{L}{j_T}}{2\left(\frac{L}{j_T}\right)^3\left(\cos \pi\frac{q}{N} - \cosh \frac{L}{j_T}\right)}\right] = 0 \qquad \text{(A16)}$$

$$(q = 1, 2, \ldots N - 1)$$

Equations (A14), (A15), and (A16) constitute the complete set of closed-form criterions for the stability of parallel tension and compression members connected by elastic deflectional springs. The correct criterion for any given values of S and L/j_T is the one which yields the lowest value of L/j_C.

Equal Tension and Compression Loads

When the tension and compression loads are equal, which would be the case if the structure were an airplane wing subjected to a bending moment, the stability criterions become

$$\frac{1}{S} - \frac{\sin \frac{L}{j_C}}{2\left(\frac{L}{j_C}\right)^3\left(\cos \pi\frac{q}{N} - \cos \frac{L}{j_C}\right)} + \frac{\sinh \frac{1}{\sqrt{r}} \frac{L}{j_C}}{\frac{2}{\sqrt{r}}\left(\frac{L}{j_C}\right)^3\left(\cos \pi\frac{q}{N} - \cosh \frac{1}{\sqrt{r}} \frac{L}{j_C}\right)} = 0 \qquad \text{(A17)}$$

$$(q = 1, 2, \ldots N - 1)$$

and

$$\frac{L}{j_C} = \pi \tag{A18}$$

$$(q = N)$$

A stability criterion for the case of an infinite number of equally spaced springs may be readily obtained as in reference 7 by expanding equation (A17) and minimizing the result with respect to $\frac{q}{N}$; this procedure gives

$$\cos \pi\frac{q}{N} = \frac{\cos \frac{L}{j_C} + \cosh \frac{1}{\sqrt{r}} \frac{L}{j_C}}{2} + \frac{S}{4\left(\frac{L}{j_C}\right)^2}\left(\frac{\sin \frac{L}{j_C}}{\frac{L}{j_C}} - \frac{\sinh \frac{1}{\sqrt{r}} \frac{L}{j_C}}{\frac{1}{\sqrt{r}} \frac{L}{j_C}}\right) \tag{A19}$$

Substitution of equation (A18) in equation (A17) yields, after some simplification,

$$S = \frac{2\left(\frac{L}{j_C}\right)^2\left(\cosh \frac{1}{\sqrt{r}} \frac{L}{j_C} - \cos \frac{L}{j_C}\right)}{\left(\sqrt{\frac{\sinh \frac{1}{\sqrt{r}} \frac{L}{j_C}}{\frac{1}{\sqrt{r}} \frac{L}{j_C}}} + \sqrt{\frac{\sin \frac{L}{j_C}}{\frac{L}{j_C}}}\right)^2} \tag{A20}$$

Equations (A18) and (A20) are the stability criterions for the case of an infinite number of equally spaced springs when the tension and compression loads are equal.

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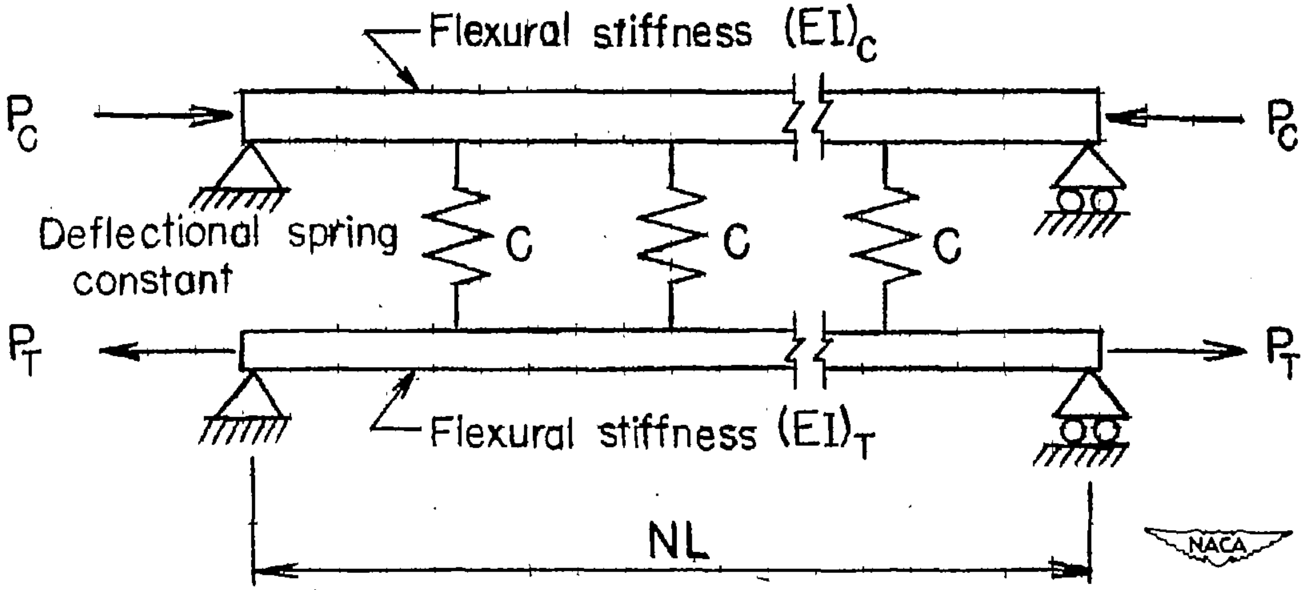

Figure 1.– Parallel tension and compression members connected by elastic deflectional springs.

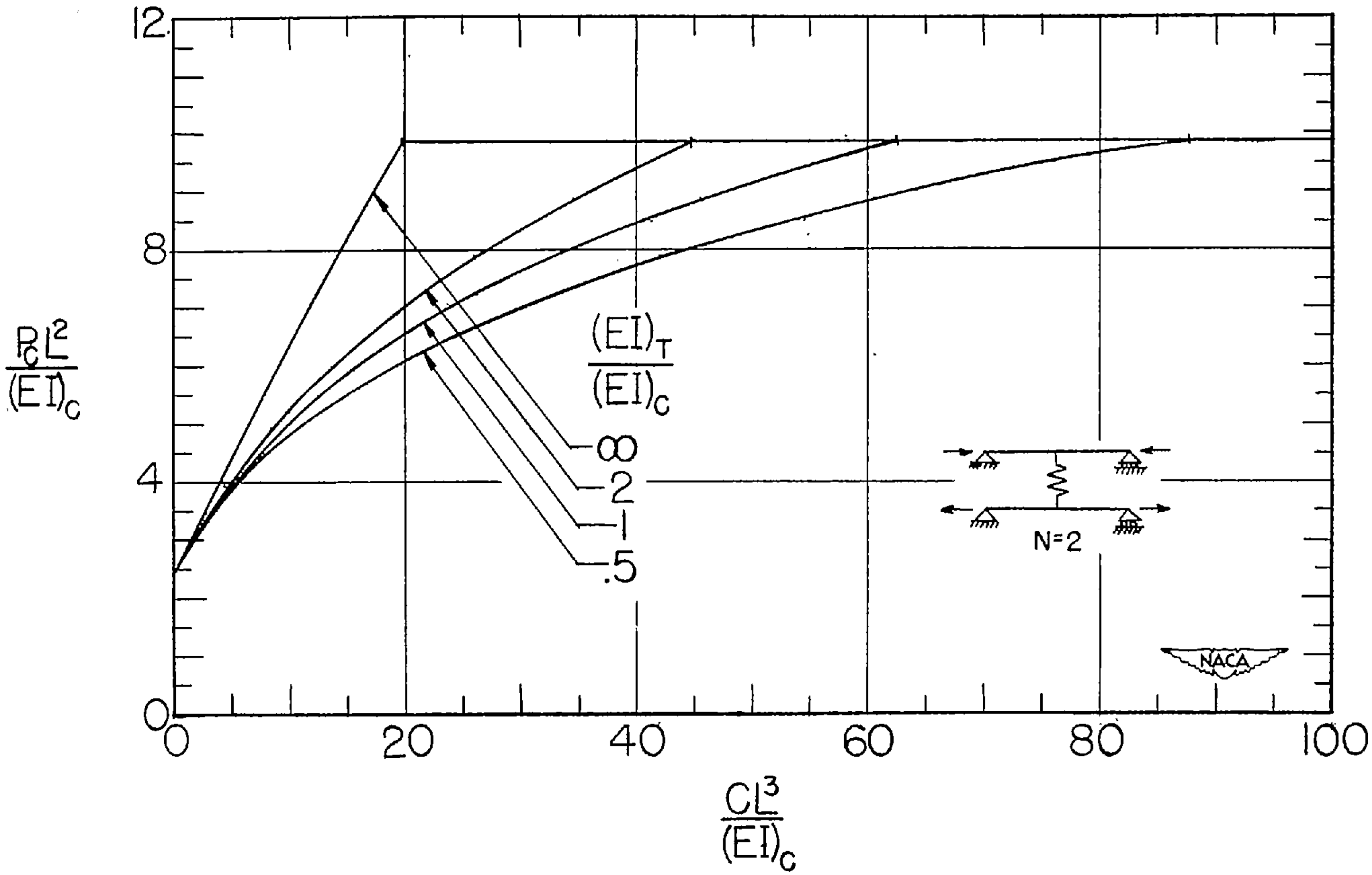

Figure 2.– Stability curves for equally loaded tension and compression members having two spans.

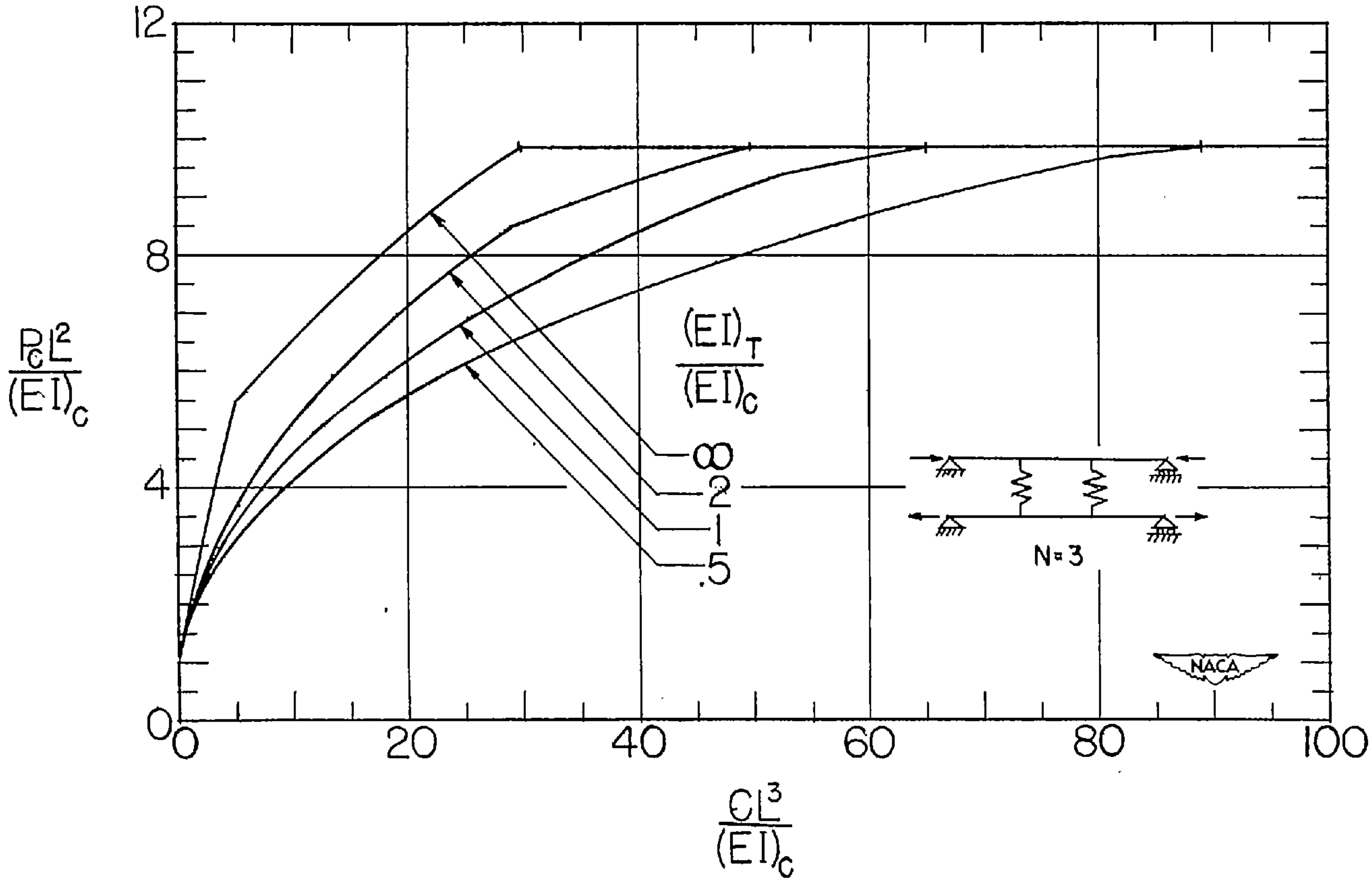

Figure 3.– Stability curves for equally loaded tension and compression members having three spans.

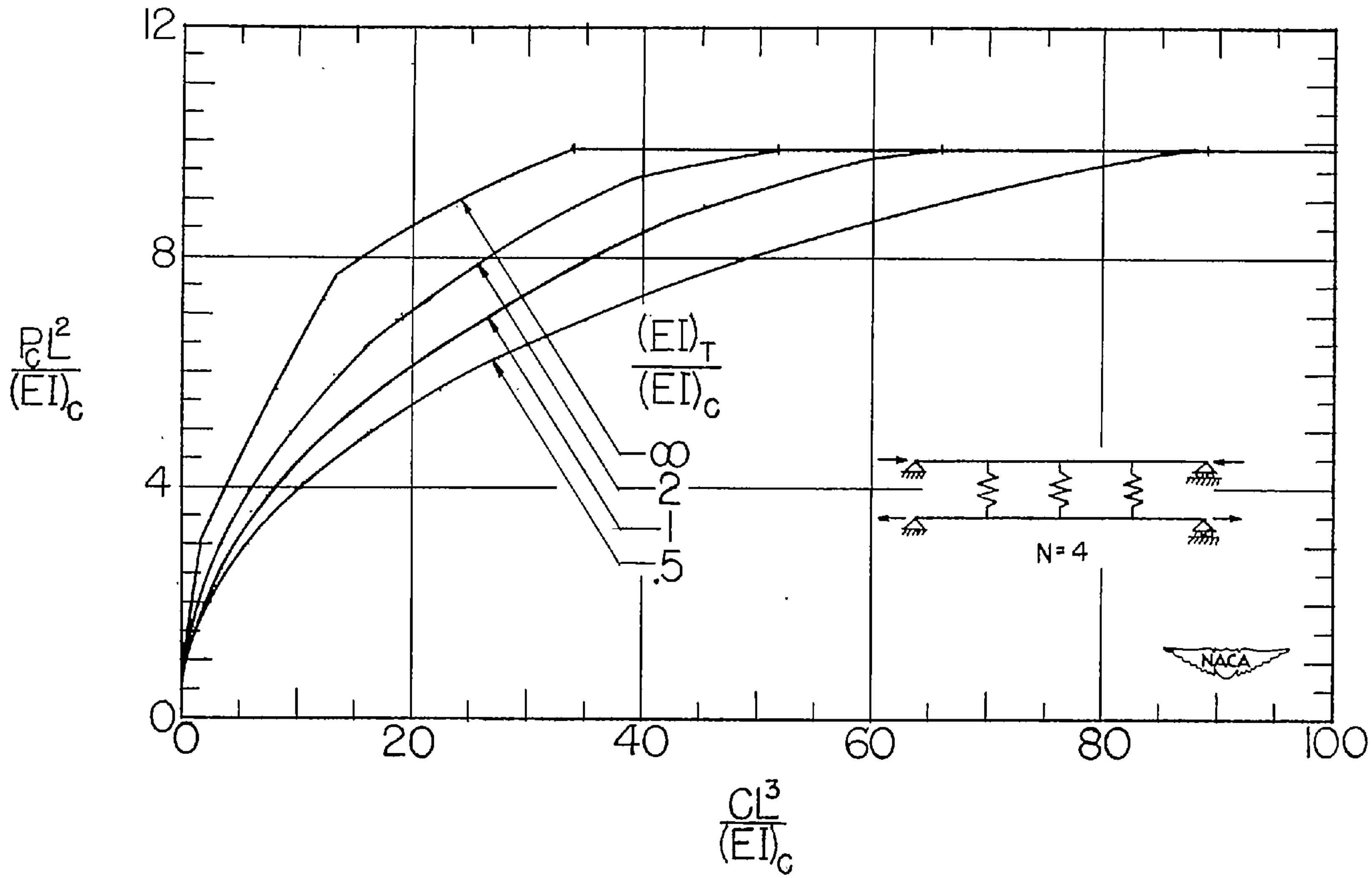

Figure 4.– Stability curves for equally loaded tension and compression members having four spans.

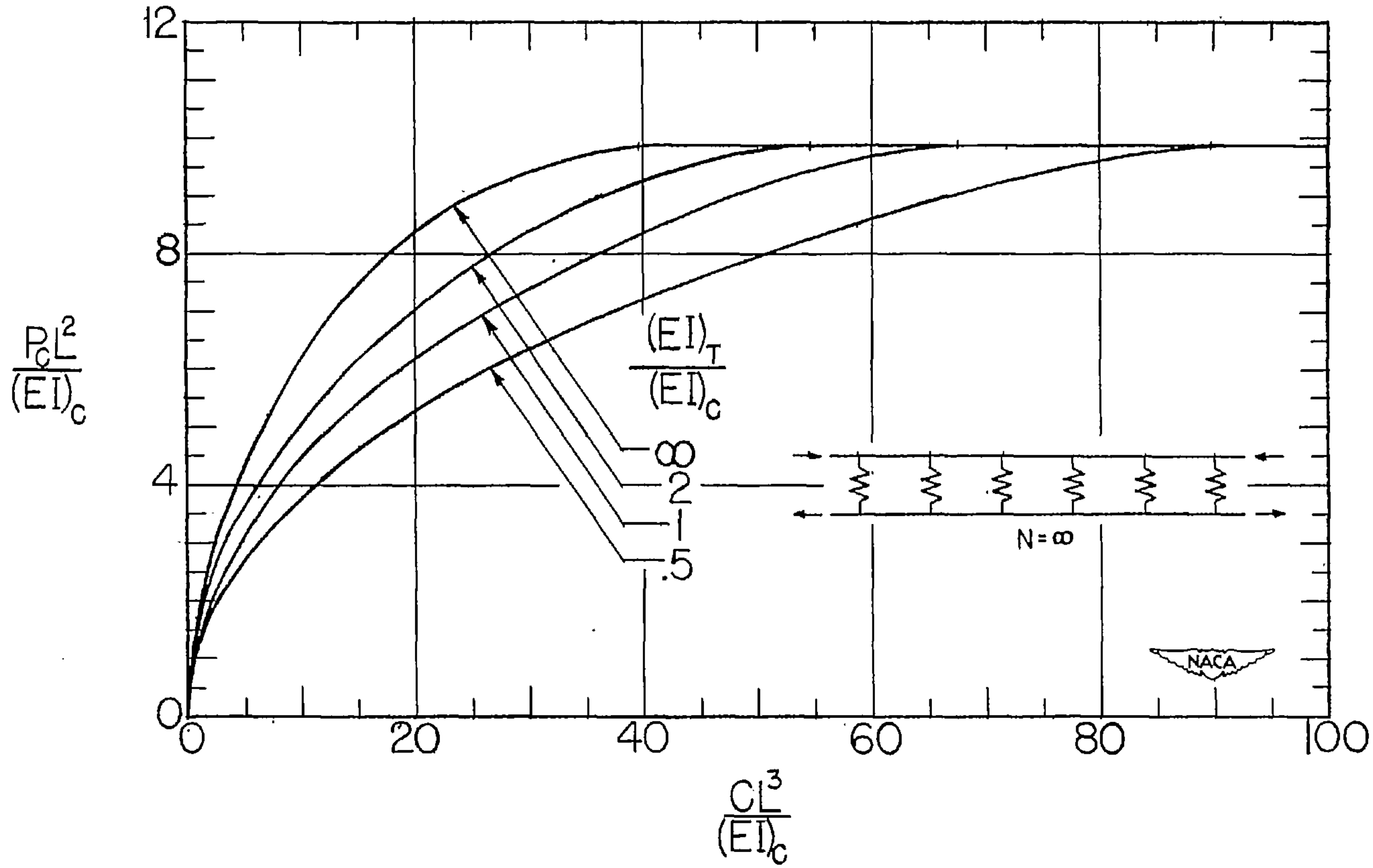

Figure 5.– Stability curves for equally loaded tension and compression members having an infinite number of spans.